Journal of Statistical Physics, Vol. 122, No. 2, January 2006 (© 2006)
DOI: 10.1007/s10955-005-8007-9

A Note on the Guerra and Talagrand Theorems for Mean Field Spin Glasses: The Simple Case of Spherical Models

Silvio Franz[1] and Francesca Tria[1]

Received May 4, 2005; accepted July 26, 2005

The aim of this paper is to discuss the main ideas of the Talagrand proof of the Parisi Ansatz for the free-energy of Mean Field Spin Glasses with a physicist's approach. We consider the case of the spherical p-spin model, which has the following advantages: (1) the Parisi Ansatz takes the simple "one step replica symmetry breaking form," (2) the replica free-energy as a function of the order parameters is simple enough to allow for numerical maximization with arbitrary precision. We present the essential ideas of the proof, we stress its connections with the theory of effective potentials for glassy systems, and we reduce the technically more difficult part of the Talagrand's analysis to an explicit evaluation of the solution of a variational problem.

KEY WORDS: Spin glasses; Mean field theory; Replica method; Interpolation method.

1. INTRODUCTION

The mathematical analysis of the low temperature mean field spin glass phase has seen enormous progresses in the last few years, after more than twenty years since the proposal of the Parisi ansatz,(1) that led to its physical understanding.(2)

The first important progress was achieved by Guerra and Toninelli,(3) who, through an interpolation among systems of different size provided a temperature independent proof of the existence of the thermodynamic limit for the free-energy and other thermodynamic quantities in a wide class of mean field spin glass systems.

[1] The Abdus Salam International Center for Theoretical Physics, Strada Costiera 11, P.O. Box 563, I-34100 Trieste, Italy; e-mail: franz@ictp.trieste.it, ftria@ictp.trieste.it

The power of the interpolating method was fully appreciated by Guerra,[4] who comparing interacting mean field systems with suitable paramagnets, was able to write the free-energy as a sum of the Parisi expression plus a remainder term, which is manifestly positive in the SK model as well as in p-spin models for even p. The method was subsequently generalized to deal with diluted spin glass models[5–7] and more recently to analyze the Kac limit of finite dimensional spin glasses.[8]

The last, but definitely not the least, step towards the proof of the Parisi free energy solution has been performed by Talagrand who, through a highly non trivial generalization of interpolation techniques to coupled replica systems, could finally prove the vanishing of the remainder in the Guerra formula.[9–11]

Unfortunately the elegance of the Talagrand approach is somewhat obscured by the mathematical necessity of dealing with many technical details in the course of the proof.

In this note we would like to present the Talagrand theorem using a physicist's approach, emphasizing the main ideas and connecting it with the theory of the glassy effective potentials discussed in the physical literature.[12–k16]

We discuss the case of the spherical p-spin model, where the saddle point equations take a particularly simple form and where it is well known that the best Parisi solution is of the simple "one step replica symmetry breaking kind" (1RSB). This allows to reduce the technically difficult part of the proof to the explicit solution of variational equations that we analyze numerically.

Our paper comprises six sections. In the second section we define the model and we set the basic definitions that we use in the rest of the paper. In the third section we illustrate the backbone of the Talagrand theorem, specializing it to our simple case. In the fourth section we derive the replica expressions and evaluate numerically suitable "effective potential functions" whose positivity implies the validity of the Parisi Ansatz. We conclude the argument in Section 5, where we show that the replica expressions for the potentials provide a lower bound for their exact values. We finally draw our conclusions.

2. SOME DEFINITIONS AND THE GUERRA FORMULA

In the spirit of discussing the results in the simplest non trivial example, we focus on the case of the spherical p-spin model.

The model considers N real spin variables σ_i, $i = 1, \ldots, N$, subject to a global spherical constraint $\sum_{i=1}^{N} \sigma_i^2 = N$, and interacting through a p-body Hamiltonian:

$$H[\sigma] = -\sum_{i_1 < i_2 < \cdots i_p}^{1,N} J_{i_1 \cdots i_p} \sigma_{i_1} \cdots \sigma_{i_p} \,. \tag{1}$$

The couplings $J_{i_1 \cdots i_p}$ are i.i.d. Gaussian random variables with zero mean and variances:

$$E_J\left[J^2_{i_1 \cdots i_p}\right] = \frac{p!}{2N^{p-1}}, \tag{2}$$

and we are not considering the case of a non zero external magnetic field.

The best Parisi solution is known to be of 1RSB kind with $q_0 = 0$. The Guerra interpolating Hamiltonian[4] to the level of 1RSB can be written as:

$$H_t[\sigma] = \sqrt{t} H[\sigma] - \sqrt{1-t} \sum_{i=1}^{N} (h_i + h_{d,i})\sigma_i, \tag{3}$$

where the interpolating parameter t runs in the interval [0, 1] and h_i and $h_{d,i}$ are Gaussian i.i.d. fields with zero mean and variances that can be written in terms of a parameter $q \in [0, 1]$ as:

$$E_h\left[h_i^2\right] = \frac{p}{2} q^{p-1} \qquad E_{h_d}\left[h_{d,i}^2\right] = \frac{p}{2}(1 - q^{p-1}) \,. \tag{4}$$

One then defines the generalized free-energy per spin:

$$f_t = -\frac{1}{\beta N m} E_J \ln E_h \left[E_{h_d} Z_t(h_i, h_{d,i})\right]^m, \tag{5}$$

where Z_t indicates the partition function at inverse temperature β computed with the Hamiltonian H_t, while m is a number in the interval (0, 1]. Notice that f_1 is the free-energy of the original model, while f_0 is the free-energy of a simple paramagnet which can be readily computed expressing the spherical constraint through a Lagrange multiplier and using standard saddle point method. This gives:

$$\begin{aligned} 2 f_0 = -\frac{1}{\beta} \min_{\lambda \in R} \Bigg\{ & \lambda - \ln\left(\lambda - \frac{\beta^2 p}{2}(1 - (1-m) q^{p-1})\right) \\ & - \frac{m-1}{m} \ln\left(\lambda - \frac{\beta^2 p}{2}(1 - q^{p-1})\right)\Bigg\}, \end{aligned} \tag{6}$$

Formula (5) makes a contact between the interpolating method and the cavity method. In fact it is well known that (5) can also be written in the form[17–19] (and for a recent review see also):[20]

$$f_t = -\frac{1}{\beta N} \mathrm{E}_{\mathrm{J},\mathrm{f}_\gamma} \ln \sum_{\gamma=1}^{\infty} w_\gamma E_d(Z_t(h_{i,\gamma}, h_{i,d})), \tag{7}$$

where $w_\gamma = \frac{e^{-\beta f_\gamma}}{\sum_{\gamma=1}^{\infty} e^{-\beta f_\gamma}}$ are random weights derived from "free-energies" f_γ that realize a Poisson point process with exponential density[2] $\rho(f) = \beta m\, e^{\beta m f}$, and the $h_{i,\gamma}$ are drawn independently for each i and γ from the Gaussian distributions defined by (4). The fields $h_{i,d}$, h_i receive then naturally the interpretation of cavity random fields, whose variances have to be determined self-consistently in order to estimate the free-energy f_1.

By simple integration by part, it is possible to see that the free-energy f_t can be written as[4]

$$f_t = f_0 + t\frac{\beta(p-1)}{4}(1-(1-m)q^p) + \int_0^t ds\, \mathcal{R}_s. \tag{8}$$

The first two terms together in (8) provide the 1RSB replica free-energy f_t^{rep} for the interpolating model. $\mathcal{R}_t$ is the t derivative of a remainder term which can be written as:

$$\mathcal{R}_t = \frac{\beta}{4}(1-m)\langle q(\sigma,\tau)^p - pq(\sigma,\tau)q^{p-1} + (p-1)q^p\,\rangle_{1,t} + \frac{\beta}{4}m\,\langle q(\sigma,\tau)^p\rangle_{0,t}, \tag{9}$$

where $q(\sigma, \tau)$ denotes the overlap function:

$$q(\sigma,\tau) = \frac{1}{N}\sum_{i=1}^{N}\sigma_i\tau_i\,. \tag{10}$$

The two averages $\langle\cdot\rangle_{1,t}$ and $\langle\cdot\rangle_{0,t}$ are defined introducing two kinds of interpolating Hamiltonians for two copies of the system with the same interactions,[10] of the form respectively:

$$H_t^{(1)}[\sigma,\tau] = \sqrt{t}H[\sigma] + \sqrt{t}H[\tau] - \sqrt{1-t}\sum_{i=1}^{N}\left(\left(h_i + h_{d,i}^1\right)\sigma_i + \left(h_i + h_{d,i}^2\right)\tau_i\right)\,, \tag{11}$$

with fields h_i and $h_{i,d}^r$, $r = 1, 2$, distributed according to (4) with $h_{i,d}^1$ and $h_{i,d}^2$ independent, and:

$$H_t^{(0)}[\sigma,\tau] = \sqrt{t}H[\sigma] + \sqrt{t}H[\tau] - \sqrt{1-t}\sum_{i=1}^{N}\left(\left(h_i^1 + h_{d,i}^1\right)\sigma_i + \left(h_i^2 + h_{d,i}^2\right)\tau_i\right), \tag{12}$$

where now, all the fields h_i^1 and h_i^2, $h_{i,d}^1$ and $h_{i,d}^2$ are drawn independently with the distribution (4).

---

[2] This means that the numbers of levels in an interval of free energies $[f_1, f_2]$ follows a Poissonian distribution with average $e^{\beta m f_2} - e^{\beta m f_1}$ and that the number of levels in disjont intervals are independent.

To enlighten the notations it is useful to define averaged partition functions:

$$\mathcal{Z}_t^{(0)}(h^1, h^2) \equiv E_{h_d^1, h_d^2}\left[Z_t^{(0)}\big(h^1, h^2, h_d^1, h_d^2\big)\right],$$
$$\mathcal{Z}_t^{(1)}(h) \equiv E_{h_d^1, h_d^2}\left[Z_t^{(1)}\big(h, h_d^1, h_d^2\big)\right], \tag{13}$$

and, for a given function of two spin configurations $k(\sigma, \tau)$, averaged Boltzmann averages:

$$\omega_1(k(\sigma,\tau)) \equiv \frac{E_{h_d^1, h_d^2}\int_{-\infty}^{+\infty}\mathcal{D}\sigma\,\mathcal{D}\tau\, k(\sigma,\tau)\exp\big(-\beta H_t^{(1)}[\sigma,\tau]\big)}{\mathcal{Z}_t^{(1)}(h)}, \tag{14}$$

$$\omega_0(k(\sigma,\tau)) \equiv \frac{E_{h_d^1, h_d^2}\int_{-\infty}^{+\infty}\mathcal{D}\sigma\,\mathcal{D}\tau\, k(\sigma,\tau)\exp\big(-\beta H_t^{(0)}[\sigma,\tau]\big)}{\mathcal{Z}_t^{(0)}(h^1, h^2)}, \tag{15}$$

$$\mathcal{D}\sigma \equiv d\sigma\,\delta\left(\sum_{i=1}^{N}\sigma_i^2 - N\right). \tag{16}$$

With these definitions, we can finally write:

$$\langle k(\sigma,\tau)\rangle_{1,t} \equiv E_J\left(\frac{E_h \mathcal{Z}_t^{(1)}(h)^{m/2}\,\omega_1(k(\sigma,\tau))}{E_h \mathcal{Z}_t^{(1)}(h)^{m/2}}\right), \tag{17}$$

$$\langle k(\sigma,\tau)\rangle_{0,t} \equiv E_J\left(\frac{E_{h_1,h_2} \mathcal{Z}_t^{(0)}(h^1,h^2)^{m}\,\omega_0(k(\sigma,\tau))}{E_{h_1,h_2} \mathcal{Z}_t^{(0)}(h^1,h^2)^{m}}\right). \tag{18}$$

It is apparent from the convexity of the function q^p for even p that both averages in the remainder (9) are non-negative.[4] Talagrand theorem consists in proving that in the thermodynamic limit, for an appropriate temperature-dependent choice of the parameters q, m, both terms in (9) are indeed equal to zero at all temperatures. For $t = 1$ the free-energy f_1 is independent of q and m. The minimization of the remainder $\int_0^1 dt\, \mathcal{R}_t$ is therefore equivalent to the maximization of the 1RSB free-energy. We will need to consider the t-dependent free-energy, for the values of q and m that achieve this maximization.

3. TALAGRAND THEOREM

The Talagrand's analysis of the remainder can be divided in three logical steps:

1. From the Taylor formula one simply observes that:

$$q(\sigma,\tau)^p - pq(\sigma,\tau)q^{p-1} + (p-1)q^p \le \frac{p(p-1)}{2}(q(\sigma,\tau)-q)^2, \quad (19)$$

and it is evident for $p > 2$ that:

$$q(\sigma,\tau)^p < q(\sigma,\tau)^2\,; \quad (20)$$

to bound the remainder it is then enough to bound the averages $\langle (q(\sigma,\tau)-q)^2\rangle_{1,t}$ and $\langle q(\sigma,\tau)^2\rangle_{0,t}$.
2. In order to get estimates of the averages $\langle (q(\sigma,\tau)-q)^2\rangle_{1,t}$ and $\langle q(\sigma,\tau)^2\rangle_{0,t}$, one considers the t-dependent probabilities $P_{j,t}(q_C) = \langle \delta(q(\sigma,\tau)-q_C)\rangle_{j,t}$ for $j = 0, 1$, that the overlap takes a value q_C, respectively in the ensembles defined by $\langle\cdot\rangle_{1,t}$ and $\langle\cdot\rangle_{0,t}$. The main part of the theorem consists in proving that, given $\epsilon > 0$, for each $t < 1$, a t-dependent positive constant $C(t)$ exists such that, if $(q_C - q)^2 \ge C(t)(f_t - f_t^{\rm rep}) + \epsilon$ and $q_C^2 \ge C(t)(f_t - f_t^{\rm rep}) + \epsilon$, then it exists a positive constant $C'(t,\epsilon)$ such that $P_{1,t}(q_C) < \exp(-C'(t,\epsilon)N)$ and $P_{0,t}(q_C) < \exp(-C'(t,\epsilon)N)$, where $f_t^{\rm rep}$ indicates the t-dependent free energy calculated with the Parisi ansatz.

   We will explicitely see in Section 4.2 that the constant $C'(t,\epsilon)$ turns out to be a monotonically decreasing function of t, behaving as $\epsilon(1-t)^2$ for t close to 1.
3. The estimates in the previous point imply, in the large N limit and fixing any $t_0 < 1$, the existence of a positive t_0-dependent constant $K(t_0)$ for which we can write the differential inequality:

$$\frac{d\big(f_t - f_t^{\rm rep}\big)}{dt} \le K(t_0)\big(f_t - f_t^{\rm rep}\big)\,, \quad (21)$$

valid for every $t \le t_0$. Since $f_{t=0} = f_{t=0}^{\rm rep}$, (3) proves $f_t \le f_t^{\rm rep}$ for every $t \le t_0$, which by continuity can be extended to $t = 1$.

While the first and third points can be easily understood, the proof of the second point requires several additional steps.

3.1. Analysis of the Probabilities $P_{j,t}(q_C)$

To estimate the probabilities $P_{j,t}(q_C)$ it is useful to relate them to suitable "potential functions" for coupled replicas, similar to the ones used in the physical literature to study off-equilibrium configurations of glassy systems.[(12–16)] The potential functions can be defined as the difference between the generalized free-energies of the interpolating system for the two replica with and without an additional constraint in the mutual overlap:

$$V_{1,t}(q_C) \equiv -\frac{2}{mN\beta}\Big[E_J \ln E_h \left(\mathcal{Z}_t^{(1)}(h)\omega_1(\delta(q(\sigma,\tau)-q_C))\right)^{m/2} - E_J \ln E_h \mathcal{Z}_t^{(1)}(h)^{m/2}\Big], \tag{22}$$

$$V_{0,t}(q_C) \equiv -\frac{1}{mN\beta}\Big[E_J \ln E_{h_1,h_2} \left(\mathcal{Z}_t^{(0)}(h^1,h^2)\omega_0(\delta(q(\sigma,\tau)-q_C))\right)^{m} - E_J \ln E_{h_1,h_2} \mathcal{Z}_t^{(0)}(h^1,h^2)^{m}\Big].$$

Notice, that, despite the dependence does not appear in the notation, the Boltzmann averages $\omega(\cdot)$ depend in (22, 23) on the same fields as the adjacent partition functions $\mathcal{Z}^{(1)}$ and $\mathcal{Z}^{(0)}$ respectively.

One can then use the property (7) and write:

$$V_{j,t}(q_C) = -\frac{T}{N} E_{J,f_\gamma} \ln \left(\frac{\sum_{\gamma=1}^{\infty} w_\gamma \mathcal{Z}_t^j(h_\gamma)\omega_j(\delta(q(\sigma,\tau)-q_C))}{\sum_{\gamma=1}^{\infty} w_\gamma \mathcal{Z}_t^j(h_\gamma)} \right), \tag{23}$$

$$P_{j,t}(q_C) = E_{J,f_\gamma} \left(\frac{\sum_{\gamma=1}^{\infty} w_\gamma \mathcal{Z}_t^j(h_\gamma)\omega_j(\delta(q(\sigma,\tau)-q_C))}{\sum_{\gamma=1}^{\infty} w_\gamma \mathcal{Z}_t^j(h_\gamma)} \right). \tag{24}$$

Again, in formulae (23, 24) it should be understood that the fields appearing implicitly in the $\omega(\cdot)$ are the same of the adjacent partition functions $\mathcal{Z}_t^j$. In the weights $w_\gamma = \frac{e^{-\beta f_\gamma}}{\sum_\gamma e^{-\beta f_\gamma}}$, the variables f_γ are chosen with exponential densities respectively $\rho_1(f) = e^{\frac{\beta m}{2} f}$ in the case labeled by 1, $\rho_0(f) = e^{\beta m f}$ in the case 0.

One can now immediately see that $P_{j,t}(q_C)$ and $V_{j,t}(q_C)$ are respectively the expectation value and a properly normalized expectation of the logarithm of the same random variable $X_{j,t}(q_C)$, where the averages are performed over the random couplings and the free-energies f_γ:

$$V_{j,t}(q_C) = -\frac{T}{N} E_{J,f_\gamma} \ln X_{j,t}(q_C); \tag{25}$$

$$P_{j,t}(q_C) = E_{J,f_\gamma} X_{j,t}(q_C). \tag{26}$$

It is possible to show, using techniques explained for example in,[21] that $1/N \ln(X_{j,t}(q_C))$ is a self-averaging quantity and the following "concentration of measure property" holds:

$$Prob\left[X_{j,t}(q_C) > e^{-N\beta V_{j,t}(q_C)+N\epsilon_1}\right] + Prob\left[X_{j,t}(q_C) < e^{-N\beta V_{j,t}(q_C)-N\epsilon_1}\right] < 2e^{-\frac{N\epsilon_1 2}{2\beta}} \tag{27}$$

for all $\epsilon_1 > 0$. Noting that $X_{j,t}(q)$ is a probability density, integrating it in q on a small interval $[q_C, q_C + \Delta q]$ and passing to the limit $\Delta q \to 0$, from a direct computation one obtains:

$$P_{j,t}(q_C) \leq 4e^{-\frac{N\epsilon_1 2}{2\beta}} + e^{-N\beta V_{j,t}(q_C)-\epsilon_1 N} + e^{-N\beta V_{j,t}(q_C)+\epsilon_1 N}\,. \tag{28}$$

We thus see that $P_{j,t}(q_C)$ is bounded by an exponentially small number in N whenever $V_{j,t}(q_C)$ is strictly positive in the limit $N \to \infty$.

The remaining part of the theorem is then devoted to showing the positivity of the two potentials $V_{j,t}(q_C)$ for every $t < 1$. To this end one would like to have a lower bound provided by the replica expressions,[3] that are the quantities that we are able to evaluate. Unfortunately, with the interpolation method we cannot directly prove the desired inequality $V_{j,t}(q_C) \geq V_{j,t}^{\text{rep}}(q_C)$, where $V_{j,t}^{\text{rep}}(q_C)$, $j = 1, 0$, indicate the replica expression of the two potentials. One can however show that the replica expression gives a lower bound for the generalized free energies $f_{j,t}(q_C) = V_{j,t}(q_C) + 2f_t$. To infer from this bound the lower bound for the potentials Talagrand found an ingenious shortcut: one first prove the existence for all $t < 1$ of a positive constant $C(t)$ such that, for all $q_C \neq q$ one has $V_{1,t}^{\text{rep}}(q_C) > \frac{2(q_C-q)^2}{C(t)}$; for q_C such that $(q_C - q)^2 > C(t)(f_t - f_t^{\text{rep}})$ one can then write the following series of inequalities:

$$V_{1,t}(q_C) > V_{1,t}^{\text{rep}}(q_C) + 2f_t^{\text{rep}}(q_C) - 2f_t > V_{1,t}^{\text{rep}}(q_C) - \frac{2(q_C - q)^2}{C(t)} > 0\,. \tag{29}$$

An analogous series of inequalities can be written for $V_{0,t}(q_C)$ for all $q_C \neq 0$ where one finds $V_{0,t}^{\text{rep}}(q_C) > \frac{2q_C^2}{C(t)}$ for some $C(t)$.

In the next section we use the standard replica approach to evaluate the two potentials $V_{j,t}^{\text{rep}}(q_C)$.

4. THE POTENTIAL FUNCTIONS IN THE REPLICA APPROACH

In order to get quick heuristic estimates of V_1 and V_0 we resort to the replica analysis, postponing to Section 5 the discussion on how the expressions we find provide free-energy lower bounds. The readers not interested to the details of the replica derivation can jump directly to Section 4.2

Before entering the discussion of the potential functions, we find it useful to illustrate the analysis of the interpolating model for a single copy of the system through the replica method. This can be done starting from expression (5), considering m as an integer and substituting the log with an n/m-th power (supposed

[3] Talagrand obtained the bound within the interpolating method. Some of the solutions he uses were previously derived in[12] with the replica method.

to be an integer):

$$f_t = -\frac{1}{\beta N m} E_J \ln E_h \left[E_{h_d} Z_t(h_i, h_{d,i})\right]^m$$
$$\to -\frac{1}{\beta N n} \ln E_J \left[E_h \left[E_{h_d} Z_t(h, h_d)\right]^m\right]^{\frac{n}{m}} \quad . \tag{30}$$

Expanding the powers one sees that we are in presence of n replicas σ^a of the original systems divided in n/m groups, each with m replicas. Together with the spin configurations the fields h are also replicated. Replicas a and b in the same group have identical fields $h_i^a = h_i^b$, while replicas a, b in different groups have statistically independent fields h_i^a and h_i^b. The Gaussian distribution of the fields is summarized in the covariance matrix:

$$E\left[h_i^a h_i^b\right] + E\left[h_{i,d}^a h_{i,d}^b\right] = H_{ab}$$
$$= \begin{cases} \mathrm{p}/2 & a = b \\ \mathrm{p}/2 q^{p-1} & a, b \text{ in the same group of } m \text{ replicas} \\ 0 & a, b \text{ in different groups} \end{cases} \tag{31}$$

At this point the calculation follows the usual rails of replica analysis of mean-field spin glasses.[2] Introducing the overlap matrix Q_{ab} one finds:

$$f_t^{\mathrm{rep}} = -\lim_{n\to 0} \frac{1}{n} S.P. \left\{ \frac{\beta t}{4} \sum_{ab} Q_{ab}^p + \frac{(1-t)\beta}{2} \sum_{a\neq b} H_{ab} Q_{ab} + \frac{1}{2\beta} \ln \det Q \right\}. \tag{32}$$

where $S.P.$ denotes saddle point evaluation over the parameters Q_{ab}. The saddle point equations for Q read:

$$\frac{p\beta t}{4} Q_{ab}^{p-1} + \frac{1-t}{2} \beta H_{ab} + \frac{1}{2\beta} (Q^{-1})_{ab} = 0. \tag{33}$$

The values of H_{ab} that make the saddle point equations (33) independent of t, are such that:

$$H_{ab} = \frac{p}{2} Q_{ab}^{p-1}\,; \tag{34}$$

$$\beta^2 \frac{p}{2} Q_{ab}^{p-1} = -(Q^{-1})_{ab}\,. \tag{35}$$

This is the choice needed to minimize the remainder (9).

The further analysis of expression (35) is standard[22] and will not be reproduced here.

We pass now to the slightly more involved expressions for the potentials $V_{j,t}(q_C)$. We give full details for the potential $V_{1,t}(q_C)$. An analogous procedure can be followed for $V_{0,t}(q_C)$. The expression for the "constrained free energy" $f_{1,t}(q_C)$ is given from (22) by:

$$
\begin{aligned}
f_{1,t}(q_C) &= -\frac{2}{mN\beta} E_J \ln E_h \left(\mathcal{Z}_t^{(1)}(h)\omega_1(\delta(q(\sigma,\tau)-q_C)) \right)^{m/2}, \\
r &= 1,2, \quad a = 1,\dots,n \qquad (36) \\
&\to -\frac{2}{nN\beta} \ln E_{J,h^{ra},h_d^{ra}} \int_{-\infty}^{+\infty} \prod_{r,a} \mathcal{D}\sigma_a^r \, E_{h_d^{ra}} \\
&\quad \times \exp\left[-\beta \sum_{r,a} H_1\left(h_d^{ra}, h^{ra}\right) \right] \prod_a \delta\left(q\left(\sigma_a^1, \sigma_a^2\right) - q_C\right).
\end{aligned}
$$

We now consider two copies, indexed by $r = 1, 2$, replicated n times. The fields $h^{r,a}$ and $h^{s,b}$ are equal if a and b belong to the same group of $\frac{m}{2}$ replica, while they are independent in the other case for all choices of r and s; the fields h_d^{ra} are independent for each replica ra. In this case we define a field covariance matrix $\mathcal{H}_{ra,sb} = E[h_i^{ra} h_i^{sb}] + E[h_{i,d}^{ra} h_{i,d}^{sb}]$ which can be conveniently recast in the form $H_{a,b} = \mathcal{H}_{ra,rb}$ ($r = 1, 2$), $\Delta_{a,b} = \mathcal{H}_{ra,sb}$ ($r \neq s = 1, 2$). The form of these matrices is given by:

$$
H_{ab} = \begin{cases} \mathrm{p}/2 & a = b \\ \mathrm{p}/2q^{p-1} & a, b \text{ in the same group of } m/2 \text{ replica} \\ 0 & a, b \text{ in different groups} \end{cases} \qquad (37)
$$

$$
\Delta_{ab} = \begin{cases} \mathrm{p}/2q^{p-1} & a = b \\ \mathrm{p}/2q^{p-1} & a, b \text{ in the same group of } m/2 \text{ replica} \\ 0 & a, b \text{ in different groups} \end{cases} \qquad (38)
$$

As for the single replica system, the best choice for the values of the parameters q and m that appear in the fields' variances is the one that maximizes the free energy of the "free" two replica system, that is, the interpolating system without the additional constraint $q_{12} = q_C$.

As explained in[12] the replica computation of the constrained free-energy is analogous to the unconstrained case, except that one needs now to introduce a replica matrix $\mathcal{Q}_{ra,sb}$ where the elements $\mathcal{Q}_{ra,ra}$ are fixed to the value q_C for all r, a. In terms of $\mathcal{Q}$ and $\mathcal{H}$ the free-energy has the form (32):

$$
\begin{aligned}
f_{1,t}^{\mathrm{rep}}(q_C) = -\lim_{n\to 0} S.P.\frac{1}{n} \Bigg[&\frac{t\beta}{4} \sum_{ra,sb} \mathcal{Q}_{ra,sb}^p \\
&+ \frac{\beta(1-t)}{2} \sum_{ra,sb} \mathcal{H}_{ra,sb}\mathcal{Q}_{ra,sb} + \frac{1}{2\beta}\mathrm{Tr}\ln \mathcal{Q} \Bigg]. \qquad (39)
\end{aligned}
$$

Supposing an ansatz such that $\mathcal{Q}_{1a,1b} = \mathcal{Q}_{2a,2b}$ and $\mathcal{Q}_{1a,2b} = \mathcal{Q}_{2a,1b}$, all to be taken as Parisi matrices, it is useful to define the submatrices $Q_{ab} = \mathcal{Q}_{1a,1b}$ and $P_{ab} = \mathcal{Q}_{1a,2b}$.

Before considering the explicit solution of (39), we just mention the difference in procedure to calculate the free energy $f_{0,t}^{\text{rep}}(q_C)$. In this case, by the definition (23), the h fields are statistically independent for the two copies, and within each copy $h^a = h^b$ if a and b belong to the same group of m replica, while h^a and h^b are independent in the other case. With the same meaning as before, we can then write:

$$H_{ab} = \begin{cases} \mathrm{p}/2 & a = b \\ \mathrm{p}/2q^{p-1} & a, b \text{ in same group of } m \text{ replica} \\ 0 & a, b \text{ in different groups} \end{cases} \tag{40}$$

$$\Delta_{ab} = 0 \quad \forall a, b. \tag{41}$$

keeping this difference in mind, the formal expression for the free energy $f_{0,t}^{\text{rep}}(q_C)$ is still given by (39).

We now want to find solutions for the matrix $\mathcal{Q}_{ra,sb}$ in both cases.

4.1. Analysis of the Saddle Point Equations

We would like now to show the existence of solutions to the saddle point equations that have respectively $V_{1,t}^{\text{rep}}(q_C) > 0$ for $q_C \neq q$ and $V_{0,t}^{\text{rep}}(q_C) > 0$ for $q_C \neq 0$.

The simplest Ansatz that we can try for the matrices $Q_{a,b}$ and $P_{a,b}$ is of 1RSB type.

$$q(x) = \begin{cases} q_0 = 0 & 0 \leq x < m_1 \\ q_1 & m_1 \leq x < 1 \end{cases} \tag{42}$$

$$p(x) = \begin{cases} p_0 = 0 & 0 \leq x < m_1 \\ p_1 & m_1 \leq x < 1 \end{cases} \quad , \tag{43}$$

As we will see below, this is a valid Ansatz giving a positive potential for the case of $V_{0,t}^{\text{rep}}(q_C)$.

Unfortunately, we numerically found that the best variational solution of this form for $f_{1,t}^{\text{rep}}(q_C)$ gives negative values of the potential in some range of q_C. A good positive solution is obtained at the level of 2RSB, which is the form we use, solving numerically the variational problem with respect to the parameter m_1,

q_1, q_2, p_1, p_2 of the functions:

$$q(x) = \begin{cases} q_0 = 0 & 0 \leq x < m/2 \\ q_1 & m/2 \leq x < m_1 \\ q_2 & m_1 \leq x < 1 \end{cases} \tag{44}$$

$$p(x) = \begin{cases} p_0 = 0 & 0 \leq x < m/2 \\ p_1 & m/2 \leq x < m_1 \\ p_2 & m_1 \leq x < 1 \end{cases} \tag{45}$$

that parameterize respectively the matrices Q and P. With both parametrizations (42, 43) and (44, 45), the saddle points in Eq. (39) turn out to be maxima with respect to all the variational parameters. The resulting expression for the potentials in terms of the variational parameters can be easily obtained substituting (44, 45) in (39); given its length we do not reproduce it here.

4.2. Numerical Evaluation of the Variational Potentials

Before analyzing the potentials for generic values of t it is useful to discuss its behavior for $t = 1$[13–15,16], where $V_{1,1}$ and $V_{0,1}$ coincide.

At temperatures higher than the "dynamical transition temperature"[23] T_d, a single, locally quadratic absolute minimum in $q = 0$ is present, with $V_{1,1}^{\mathrm{rep}}(0) \equiv V_{1,1}^{\mathrm{rep}}(0) = 0$. When the temperature is lowered below T_d but is still higher than the thermodynamical transition temperature T_s, besides the $q = 0$, $V_{1,1}^{\mathrm{rep}}(0) = 0$, a second local minimum, higher than the previous one appears for a value of $q > 0$. When T reaches T_s and for values of $T < T_s$, the replica symmetry is broken, the two minima are degenerate $V_{1,1}^{\mathrm{rep}}(0) = V_{1,1}^{\mathrm{rep}}(q) = 0$ and the argument of the high overlap minimum q equals to the solution of the 1RSB variational problem at that temperature.

The behavior of the $t < 1$ replica potentials is investigated solving numerically the variational equations with respect to all the variational parameters.

Let us first discuss the simple case of the high temperature replica symmetric phase for $T > T_s$. In this case only the $V_{0,t}^{\mathrm{rep}}$ potential has to be considered and we show in Fig. 1 that the required inequality $V_{0,t}^{\mathrm{rep}}(q_C) > 0$ for $q_C \neq 0$, already satisfied for $t = 1$, remains valid for every $t < 1$.

Let us then analyze the RSB phase ($T \leq T_s$) when $t < 1$ and show that the minimum in zero (case 1) or in q (case 0) becomes respectively higher than the absolute one.

We show in Fig. 2 the results of the saddle point evaluation of the potential $V_{0,t}^{\mathrm{rep}}(q_C)$ for different values of the parameter t at a fixed temperature $T < T_s$, having verified that the behavior does not change for different values of temperature below the static transition.

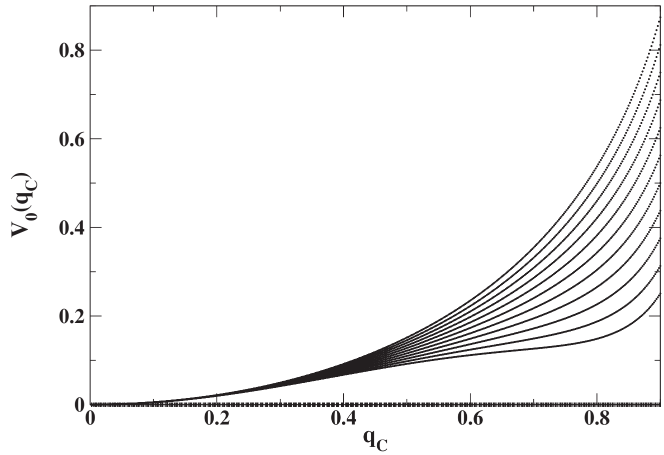

Fig. 1. The potential $V^{\rm rep}_{0,t}(q_C)$ for different values of t at the temperature T, $T_s = 0.503 < T = 0.526 < T_d = 0.544$, for $p = 4$. From the bottom to the top the parameter t runs from 1 to 0, decreasing by 0.1 at each plot. We have plotted only the results for $q_C \geq 0$, being the potential symmetric for negative values of q_C.

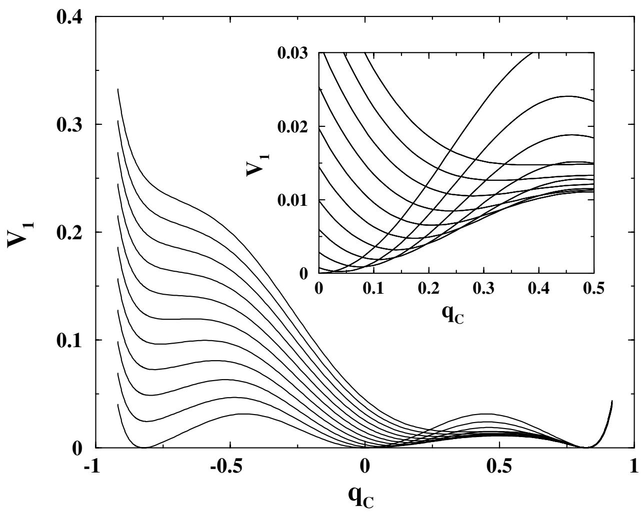

Fig. 2. The potential $V^{\rm rep}_{1,t}(q_C)$ at the inverse temperature $\beta = 3$ and $p = 4$. From the bottom to the top the parameter t runs from 1 to 0, decreasing by 0.1 at each plot. In the zoom is shown the potential at the same inverse temperature near the higher minimum, for t varying from 1 (at the bottom) to 0.9 (at the top) decreasing by 0.01 at each plot. Note that the potential is not symmetric with respect to positive and negative values of q_C, being the minimum at $q_{12} = 0$.

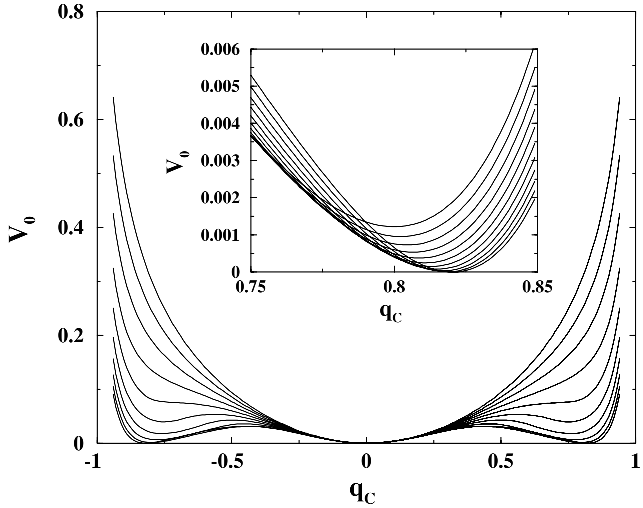

Fig. 3. The potential $V_{0,t}^{\text{rep}}(q_C)$ at the inverse temperature $\beta = 3$ and $p = 4$. From the bottom to the top the parameter t runs from 1 to 0.9, decreasing by 0.0 at each plot. In the zoom is shown the potential at the same inverse temperature near the higher minimum at the same inverse temperature and for the same values of the parameter t. Note that the potential is now symmetric with respect to positive and negative values of q_C.

In Fig. 3 we plot the potential $V_{1,t}^{\text{rep}}(q_C)$ for different values of the parameter t at the same temperature $T < T_s$ as in Fig. 3.

The two potentials $V_{1,t}^{\text{rep}}(q_C)$ and $V_{0,t}^{\text{rep}}(q_C)$ are greater than zero for every value of q_C different, respectively, from q and zero, and are quadratic near the (lower) minimum.

We numerically find a quadratic dependence on $1 - t$, for $t \simeq 1$, of the value of the potential in the higher local minimum (Figs. 4 and 5), implying that the constant $1/C(t)$ behaves as $(1 - t)^2$ for t close to one.

5. LOWER BOUND FOR THE INTERPOLATING FREE ENERGY

To conclude the argument we now show that the constrained free-energies calculated in the previous section within the replica approach indeed supply lower bounds for the true ones.

We just repeat here the argument given by Talagrand, specializing to our simple case. For each value of t we define an interpolating Hamiltonian H_u that relates, varying the parameter u in the interval [0, 1], the two replica interpolating system with a suitable paramagnet.

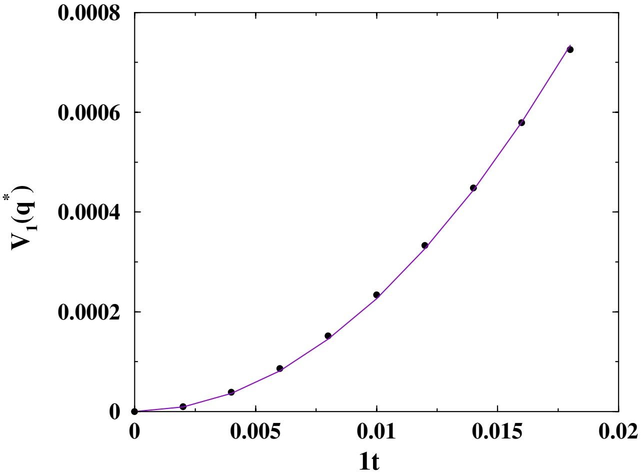

Fig. 4. Value of the potential $V_{1,t}^{\rm rep}$ in the higher local minimum, as a function of $1-t$, at the inverse temperature $\beta = 3$ and $p = 4$. The continuous line is a quadratic fit with the function $f(x) = ax^2$, $a = 2.26684$. (Color online).

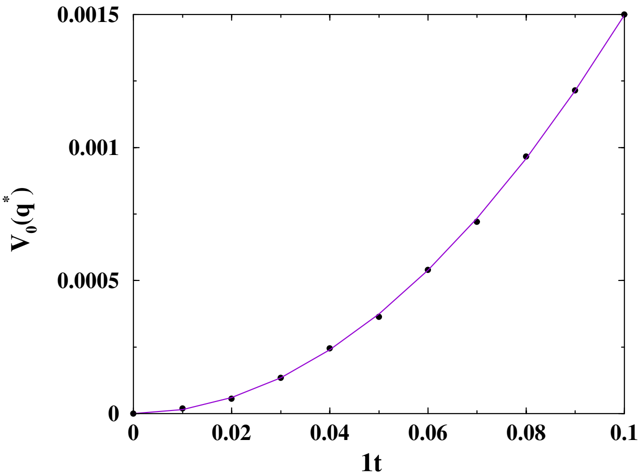

Fig. 5. Value of the potential $V_{0,t}^{\rm rep}$ in the higher local minimum, as a function of $1-t$, at the inverse temperature $\beta = 3$ and $p = 4$. The continuous line is a quadratic fit with the function $f(x) = ax^2$, $a = 0.149782$. (Color online).

Once again, we only discuss the case of $f_{1,t}(q_C)$, the same procedure being valid with minor modifications for $f_{0,t}(q_C)$.

We have found in the previous section a good positive solution at the level of 2RSB; to deal with this case within an interpolating scheme we need three families of fields for each copy. One then defines:

$$H_u[\sigma,\tau] = \sqrt{ut}H_p[\sigma] + \sqrt{ut}H_p[\tau] - \sqrt{1-t}\sum_{i=1}^{N}\Big(\big(h_i + h^1_{d,i}\big)\sigma_i + \big(h_i + h^2_{d,i}\big)\tau_i\Big) - \sqrt{(1-u)t}\sum_{i=1}^{N}\Big(\big(g^1_{1,i} + g^1_{2,i} + g^1_{d,i}\big)\sigma_i + \big(g^2_{1,i} + g^2_{2,i} + g^2_{d,i}\big)\tau_i\Big), \tag{46}$$

where the fields h and h_d are defined as in (4) and h^1_d and h^2_d chosen independently, while the variances and the covariances of the fields g can be written, in a notation consistent with Section 4, as:

$$E_g\left[\left(g^r_1\right)^2\right] = \frac{p}{2}q_1^{p-1} \quad E_g\left[\left(g^r_2\right)^2\right] = \frac{p}{2}q_2^{p-1} \quad E_g\left[\left(g^r_d\right)^2\right] = \frac{p}{2}\left(1-q_2^{p-1}\right),$$
$$r = 1,2 \quad E_g\left[g^1_1 g^2_1\right] = \frac{p}{2}p_1^{p-1} \quad E_g\left[g^1_2 g^2_2\right] = \frac{p}{2}p_2^{p-1}$$
$$E_g\left[g^1_d g^2_d\right] = \frac{p}{2}\left(q_C^{p-1} - p_2^{p-1}\right). \tag{47}$$

In a completely analogous way as in (5), we define the free energy:

$$\tilde{f}_u = -\frac{2}{\beta N m} E_J \ln\left[E_{g^1_1,g^2_1,h}\left(E_{g^1_2,g^2_2}\mathcal{Z}_{C,u}{}^{m_1}\right)^{m/2m_1}\right], \tag{48}$$

$$\mathcal{Z}_{C,u} = \int_{-\infty}^{+\infty}\mathcal{D}\sigma\,\mathcal{D}\tau\,E_{h^1_d,h^2_d,g^1_d,g^2_d}\delta(q(\sigma,\tau)-q_C)\exp -\beta H_u[\sigma,\tau],$$

where it is evident that $\tilde{f}_{u=1,t} = f_t(q_C)$. On the other hand, one can check that $\tilde{f}_{u=0,t}$ reproduces the non trivial part of the replica expression of $f_t(q_C)$ and one has

$$f_t^{\text{rep}}(q_C) = \tilde{f}_{u=0,t} + \frac{t\beta(p-1)}{2}\left\{1 - (1-m_1)q_2^p - \left(m_1 - \frac{m}{2}\right)q_1^p + q_C^p - (1-m_1)p_2^p - \left(m_1 - \frac{m}{2}\right)p_1^p\right\}, \tag{49}$$

when the parameters that appear in the variances (47) maximize the expression (49) for $t = 1$.

We can thus repeat the argument as for the "standard" interpolating method: by the usual integrations by part one sees that the derivative $\frac{d\tilde{f}_u}{du}$ can be written, for every t, as:

$$\frac{d\tilde{f}_u}{du} = \frac{t\beta(p-1)}{2}\left\{1-(1-m_1)q_2^p - \left(m_1 - \frac{m}{2}\right) q_1^p + q_C^p - (1-m_1)p_2^p - \left(m_1 - \frac{m}{2}\right) p_1^p\right\} + \mathcal{R}_u, \tag{50}$$

where $\mathcal{R}_u$ is a remainder with the explicit expression:

$$\begin{aligned}\mathcal{R}_u = {} & \frac{t\beta}{2}(1-m_1)\langle 2q(\sigma,\tau)^p - pq(\sigma,\tau)\big(q_2^{p-1}+p_2^{p-1}-q_1^{p-1}-p_1^{p-1}\big) \\ & +(p-1)\big(q_2^p+p_2^p-q_1^p-p_1^p\big)\rangle_{2,u} + \frac{t\beta}{2}\big(m_1-\frac{m}{2}\big)\langle 2q(\sigma,\tau)^p \\ & - pq(\sigma,\tau)\big(q_1^{p-1}+p_1^{p-1}\big) + (p-1)\big(q_1^p+p_1^p\big)\rangle_{1,u} + t\beta\frac{m}{2}\langle q(\sigma,\tau)^p\rangle_{0,u}.\end{aligned} \tag{51}$$

The inequality $\tilde{f}_{u=0} = f_{1,t}^{\mathrm{rep}}(q_C) \leq f_{1,t}(q_C) = \tilde{f}_{u=1}$ follows from the positivity of $\mathcal{R}_u$, that is evident from the convexity of the function q^p for even p, in the same way as in (9). In the Appendix we give the expressions for the averages that appear in the remainder.

6. CONCLUSION

The main point of this paper is to present the Talagrand proof of the Parisi anzatz in disordered mean field models in the physicist's language. With the aim of emphasizing the conceptual aspects of the proof, we have specialized it to the case of the spherical p-spin model, where the simplicity of the Parisi solution allows a great reduction of technical difficulties. The main advantages of our approach is that we circumvent the most involved part of the demonstration by a direct solution of a variational problem, which in the case faced in this paper allows for an easy numerical solution. In doing that, we emphasize the connections of the Talagrand proof with the well known method of coupled replicas and the construction of glassy potential functions previously known in the physical literature [(12)]. Though our paper does not really contain new mathematical results, we hope that it could be useful to spread the Talagrand's results in a larger community.

One of the difficulties of the Talagrand paper is that the proof was provided in a general Parisi scheme with arbitrary steps of RSB. This is needed in the case of the SK model or in the low temperature phase of the Ising p-spin model. There, the derivation is complicated by the need to go to the limit of continuous replica symmetry breaking. In particular, Talagrand showed that, for every $t < 1$, it is possible to consider a number $k = k(t)$ of steps of RSB such that an interpolating system with $j > k$ steps is solved by the Parisi Ansatz with j steps. One then has to show the positivity of j different potential functions. An alternative is to work directly in the continuous RSB limit, and to look for a lower bound for a suitable family of potential functions depending on a continuous index. These potentials can be computed explicitly in spherical models with full RSB,[12] and we hope to report soon on that case.

Thanks to the more physical approach followed in this revisitation of the Talagrand proof, it can also be easier to analyze more involved mean field systems, like those diluted, where a lower bound for the free energy was recently found[5,6] by an extension of the interpolating procedure.

APPENDIX

Generalizing the procedure used to define the averages (17), the first step to define $\langle\cdot\rangle_{0,u}$, $\langle\cdot\rangle_{1,u}$ and $\langle\cdot\rangle_{2,u}$ is to consider three four-replica Hamiltonian with the general form:

$$\begin{aligned}\mathbf{H} = {} & \sqrt{ut}H_p[\sigma] + \sqrt{ut}H_p[\tau] - \sqrt{1-t}\sum_{i=1}^{N}\left(\left(h_i + h^1_{d,i}\right)\sigma_i + \left(h_i + h^2_{d,i}\right)\tau_i\right) \\ & - \sqrt{(1-u)t}\sum_{i=1}^{N}\left(\left(g^1_{1,i} + g^1_{2,i} + g^1_{d,i}\right)\sigma_i + \left(g^2_{i,1} + g^2_{i,2} + g^2_{d,i}\right)\tau_i\right) \\ & + \sqrt{ut}H_p[\sigma'] + \sqrt{ut}H_p[\tau'] - \sqrt{1-t}\sum_{i=1}^{N}\left(\left(\tilde{h}_i + \tilde{h}^1_{d,i}\right)\sigma'_i + \left(\tilde{h}_i + \tilde{h}^2_{d,i}\right)\tau'_i\right) \\ & - \sqrt{(1-u)t}\sum_{i=1}^{N}\left(\left(\tilde{g}^1_{1,i} + \tilde{g}^1_{2,i} + \tilde{g}^1_{d,i}\right)\sigma'_i + \left(\tilde{g}^2_{i,1} + \tilde{g}^2_{i,2} + \tilde{g}^2_{d,i}\right)\tau'_i\right), \end{aligned} \tag{52}$$

where the new introduced fields have the same distribution of the corresponding h and g fields: $\tilde{h}$ have the same distribution of h, and $\tilde{g}$ the same distribution of g.

The three different Hamiltonians are specified by:

1. $\mathbf{H_0}$: the $\tilde{h}$ and $\tilde{g}$ fields are respectively independent of the fields h and g.
2. $\mathbf{H_1}$: $\tilde{g}^1_1 = g^1_1$, $\tilde{g}^2_1 = g^2_1$ and $\tilde{h} = h$, while the other $\tilde{h}$ and $\tilde{g}$ fields are independent of the corresponding h and g fields.
3. $\mathbf{H_2}$: $\tilde{g}^1_1 = g^1_1$, $\tilde{g}^2_1 = g^2_1$, $\tilde{h} = h$, $\tilde{g}^1_2 = g^1_2$, $\tilde{g}^2_2 = g^2_2$; the "annealed" fields $h^r_{i,d}$, $\tilde{h}^r_{i,d}$, $g^r_{i,d}$ and $\tilde{g}^r_{i,d}$ ($r = 1, 2$) are always independent.

We can now define, as done in (14), the Boltzmann averages of a given function of two spin configurations $k(\sigma,\tau)$, averaged over all the annealed fields $h^r_{i,d}$, $\tilde{h}^r_{i,d}$, $g^r_{i,d}$, $\tilde{g}^r_{i,d}$, $r=1,2$:

$$\Omega(k(\sigma,\tau))_{2,1,0} \equiv \frac{\int_{-\infty}^{+\infty} E_{h^r_{i,d},\tilde{h}^r_{i,d},g^r_{i,d},\tilde{g}^r_{i,d}} \mathcal{D}\sigma\, \mathcal{D}\tau\, k(\sigma,\tau)\exp\left(-\beta \mathbf{H_{2,1,0}}\right)}{\mathcal{Z}_{2,1,0}}$$

$$r=1,2,\ \mathcal{Z}_{2,1,0} \equiv \int_{-\infty}^{+\infty} E_{h^r_{i,d},\tilde{h}^r_{i,d},g^r_{i,d},\tilde{g}^r_{i,d}} \mathcal{D}\sigma\, \mathcal{D}\tau\, \delta(q(\sigma,\sigma') - q_C)\delta(q(\tau,\tau') - q_C)\exp\left(-\beta \mathbf{H_{2,1,0}}\right). \tag{53}$$

The three averages that appear in the remainder are then defined by:

$$\langle k(\sigma,\tau)\rangle_{2,u} \equiv E_J \frac{E_{g_1^1,g_1^2,h}\left[\left(E_{g_2^1,g_2^2}\mathcal{Z}_2^{m_1/2}\right)^{m/2m_1-1} E_{g_2^1,g_2^2}\mathcal{Z}_2{}^{m_1/2}\Omega(k(\sigma,\tau))_2\right]}{E_{g_1^1,g_1^2,h}\left(E_{g_2^1,g_2^2}\mathcal{Z}_2{}^{m_1/2}\right)^{m/2m_1}}; \tag{54}$$

$$\langle k(\sigma,\tau)\rangle_{1,u} \equiv E_J \frac{E_{g_1^1,g_1^2,h}\left[\left(E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_1{}^{m_1}\right)^{(m/4m_1)-1} E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_1{}^{m_1}\Omega(k(\sigma,\tau))_1\right]}{E_{g_1^1,g_1^2,h}\left(E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_1{}^{m_1}\right)^{m/4m_1}}; \tag{55}$$

$$\langle k(\sigma,\tau)\rangle_{0,u} \equiv E_J \frac{E_{g_1^1,g_1^2,h,\tilde{g}_1^1,\tilde{g}_1^2,\tilde{h}}\left[\left(E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_0{}^{m_1}\right)^{(m/2m_1)-2} E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_0{}^{m_1}\Omega(k(\sigma,\tau))_0\right]}{E_{g_1^1,g_1^2,h,\tilde{g}_1^1,\tilde{g}_1^2,\tilde{h}}\left(E_{g_2^1,g_2^2,\tilde{g}_2^1,\tilde{g}_2^2}\mathcal{Z}_0{}^{m_1}\right)^{m/2m_1}}. \tag{56}$$

REFERENCES

1. G. Parisi, *Phys. Lett.* A **73**(**3**):203–205, (1979).
2. M. Mezard, G. Parisi and M. A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
3. F. Guerra and F. L. Toninelli, *Commun. Math. Phys.* **230**:71–79 (2002).
4. F. Guerra, *Comm. Math. Phys.* **233**:1–12 (2003).
5. S. Franz and M. Leone, *J. Stat. Phys.* **111**:535 (2003).
6. S. Franz, M. Leone and F. L. Toninelli, *J. Phys. A* **36**:10967 (2003).
7. S. Franz and F. L. Toninelli, *J. Phys. A: Math. Gen.* **37**:7433–7446 (2004).
8. S. Franz and F. L. Toninelli, *J. Stat. Mech.* P01008 (2005).

9. M. Talagrand, *C. R. Acad. Sci. Paris, Ser. I* **337**:111–114 (2003).
10. M. Talagrand, *Annals of Mathematics*, to appear.
11. M. Talagrand, *Proba. Theor. Related Fields*, to appear.
12. S. Franz, G. Parisi and M. A. Virasoro, *J. Phys I* **2**(10):1869–1880 (1992).
13. S. Franz and G. Parisi, *J. Phys. I Franz* **5**:1401–1415, (1995).
14. S. Franz, M. Cardenas and G. Parisi, *J. Chem. Phys.* **110**:1726 (1999).
15. S. Franz and G. Parisi, *Phys. Rev. Lett.* **79**(13): 2486–2489 (1997).
16. G. Parisi, G. Ruocco and F. Zamponi, *Phys. Rev. E* **69**:061505 (2004).
17. M. Aizenman, R. Sims and S. L. Starr, *Phys. Rev. B* **68**:214403 (2003).
18. M. Mezard, G. Parisi and M. A. Virasoro, *Europhys Lett.* **1**:77–82 (1986); M. Mezard and G. Parisi, *Eur. Phys. J. B* **20**:217–233 (2001).
19. D. Ruelle Commun. Math. Phys. **108**:225–239 (1987).
20. F. Guerra, cond-mat/0410435 (Oct 2004).
21. M. Talagrand *Spin glasses: A challenge for mathematicians. Mean field models and cavity method* (Springer-Verlag, Berlin, 2003).
22. A. Crisanti and H. J. Sommers, *Z. Phys B - Condensed Matter* **87**:341–354 (1992).
23. L. F. Cugliandolo and J. Kurchan, *Phys. Rev. Lett.* **71**:173 (1993).
24. T. M. Nieuwenhuizen, *Phys. Rev. Lett.* **74**(21):4289–4292 (1994).